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A Contribution to the Rainich Theory of the Neutrino Field II

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Abstract

Conditions are given on the metric of a space-time for it to admit a neutrino-gravitational field. The method described involves comparing the spin coefficients of a suitably defined tetrad of null vectors. It is used to obtain the class of space-times which admits a neutrino field or a non-null electromagnetic field. The classes of space-time which admit two different neutrino fields are also described.

1. *Introduction*

We are concerned here with the questions posed by Wheeler's (1962) theory of geometrodynamics. Namely, (a) can we find conditions on the metric of a space-time for it to admit a neutrino-gravitational field? and, (b) is the neutrino field uniquely determined by the space-time? Or, under what conditions may two distinct neutrino fields be described by the same metric? A complete answer is here given to both of these questions. However, the methods are noticeably different from those used in the familiar Rainich-Misner-Wheeler theory of the 'already unified' theory of gravitation and electromagnetism. The conditions given here involve certain concomitants of the Ricci tensor rather than the Ricci tensor itself. This approach has already been applied successfully to electromagnetic fields by Ludwig (1970). He has found necessary and sufficient conditions that a space-time admit an electromagnetic field, null or non-null.

A neutrino field will be defined by a two-component spinor ξ_A satisfying Weyl's equation

$$\sigma^{\alpha}{}_{A\dot{B}}\xi^{A}{}_{;\alpha} = 0 \qquad \text{or} \qquad \xi^{A}{}_{;A\dot{B}} = 0 \tag{1.1}$$

It has a flux vector

$$l_{\mu} = \xi_{A}\,\sigma_{\mu}{}^{A\dot{B}}\xi_{\dot{B}}$$

and energy-momentum tensor

$$E_{\mu\nu} = i\{\sigma_{\mu A\dot{B}}(\xi^{A}\xi^{\dot{B}}{}_{;\nu} - \xi^{\dot{B}}\xi^{A}{}_{;\nu}) - \sigma_{\nu A\dot{B}}(\xi^{A}\xi^{\dot{B}}{}_{;\mu} - \xi^{\dot{B}}\xi^{A}{}_{;\mu})\} \tag{1.2}$$

If ξ_A is taken as a basis, a second basis spinor χ_A can be defined such that

$$\xi_A \chi^B - \chi_A \xi^B = \delta_A{}^B$$

and a tetrad of null vectors may be defined by

$$l_\mu = \xi_A \sigma_\mu{}^{A\dot{B}} \xi_{\dot{B}}, \qquad n_\mu = \chi_A \sigma_\mu{}^{A\dot{B}} \chi_{\dot{B}}, \qquad m_\mu = \xi_A \sigma_\mu{}^{A\dot{B}} \chi_{\dot{B}}$$

The spin coefficients of this 'neutrino tetrad' will be used in the notation of Newman & Penrose (1962). In this way expression (1.2) may be written as

$$\begin{aligned} E_{\mu\nu} = 2i\{(\bar{\gamma} - \gamma) l_\mu l_\nu &+ (\alpha - \bar{\beta} - \bar{\tau}) l_{(\mu} m_{\nu)} - (\bar{\alpha} - \beta - \tau) l_{(\mu} \bar{m}_{\nu)} \\ &+ (\bar{\epsilon} - \epsilon) l_{(\mu} n_{\nu)} + (\bar{\rho} - \rho) m_{(\mu} \bar{m}_{\nu)} + \bar{\sigma} m_\mu m_\nu - \sigma \bar{m}_\mu \bar{m}_\nu \\ &+ \kappa n_{(\mu} \bar{m}_{\nu)} - \bar{\kappa} n_{(\mu} m_{\nu)}\} \end{aligned} \tag{1.3}$$

and the Weyl equation (1.1) becomes

$$\epsilon = \rho, \qquad \beta = \tau \tag{1.4}$$

The following geometrical interpretation of the spin coefficients will be referred to. If $\kappa = 0$ then l_μ defines a geodesic null congruence with expansion, twist and shear proportional to Re ρ, Im ρ and $|\sigma|$ respectively. In the null congruence defined by the vector n_μ the spin coefficients $-\nu, -\mu, -\lambda$ correspond to κ, ρ, σ respectively.

Rainich type conditions for some classes of neutrino-gravitational field have already been given by Griffiths & Newing (1972) and Collinson & Shaw (1972) The results given here are general in the sense that no further restrictions on the neutrino field are made, such as energy conditions (Griffiths & Newing, 1971a; Wainwright, 1971) or causality (Audretsch, 1971). The class of space-times which admit either a neutrino field or an electromagnetic field is also mentioned. For convenience the question of uniqueness will be covered first.

2. *The Uniqueness of the Neutrino Field*

It has been shown previously (Griffiths & Newing, 1971b) that if both the energy-momentum tensor and the flux vector are known then the neutrino field is uniquely determined up to an arbitrary constant phase factor. For a Rainich type theory however only the energy-momentum tensor is known through Einstein's equations, and it must still be shown how the flux vector can be defined by the geometry. The possibility that a particular metric can describe different neutrino fields with different flux vectors must first be considered. Two possible cases arise.

First there is the possibility that a space-time will admit two neutrino fields with flux vectors in the same direction. If the two fields are defined by the spinors ξ_A and $e^\Phi \xi_A$ then the neutrino equations require that

$$\epsilon = \rho, \qquad \beta = \tau$$

$$D\phi = 0, \qquad \delta\phi = 0$$

and, since the energy-momentum tensor must be the same for the two fields it must take the form

$$E_{\mu\nu} = Al_\mu l_\nu + Bl_{(\mu}m_{\nu)} + \bar{B}l_{(\mu}\bar{m}_{\nu)}$$

and the following conditions must be satisfied

$$A = 2i(\bar{\gamma} - \gamma), \qquad B = 2i(\alpha - \bar{\beta} - \bar{\tau})$$
$$\rho = \bar{\rho}, \qquad \sigma = 0, \qquad \kappa = 0$$
$$\Delta(\phi - \bar{\phi}) = (\gamma - \bar{\gamma})(e^{-\phi-\bar{\phi}} - 1)$$
$$\bar{\delta}\phi = (\alpha - 2\bar{\tau})(e^{-\phi-\bar{\phi}} - 1)$$

The integrability conditions for these equations can easily be obtained using the Newman–Penrose methods, however, they do not lead to any immediate algebraic simplification. It can be seen, however, that in this case the neutrino flux vectors define a shear-free and twist-free null geodesic congruence.

We now investigate the second possibility that a space-time admit two neutrino fields whose flux vectors point in different directions. It is convenient to choose the basis such that ξ_A defines the first field and $e^\phi \chi_A$ defines the second. The two flux vectors are then l_μ and $e^{\phi+\bar{\phi}}n_\mu$. The neutrino equations and energy-momentum tensor of the first field are given by (1.4) and (1.3), and those of the second field are

$$\bar{\delta}\phi = \alpha - \pi, \qquad \Delta\phi = \gamma - \mu$$

and

$$\begin{aligned} E_{\mu\nu} = 2i\, e^{\phi+\bar{\phi}}\{(\epsilon - \bar{\epsilon})n_\mu n_\nu + (\bar{\alpha} - \beta + \bar{\pi})n_{(\mu}\bar{m}_{\nu)} - (\alpha - \bar{\beta} + \pi)n_{(\mu}m_{\nu)} \\ + (\gamma - \bar{\gamma})l_{(\mu}n_{\nu)} + (\mu - \bar{\mu})m_{(\mu}\bar{m}_{\nu)} - \bar{\lambda}\bar{m}_\mu\bar{m}_\nu + \lambda m_\mu m_\nu \\ - \nu l_{(\mu}m_{\nu)} + \bar{\nu}l_{(\mu}\bar{m}_{\nu)} + n_{(\mu}\bar{\phi}_{,\nu)} - n_{(\mu}\phi_{,\nu)}\} \end{aligned} \tag{2.1}$$

Since the energy-momentum tensors must be the same through Einstein's equations we may equate the coefficients of (1.3) and (2.1) giving

$$\begin{aligned} \bar{\gamma} - \gamma &= 0 \\ \alpha - \bar{\beta} - \bar{\tau} &= -\nu\, e^{\phi+\bar{\phi}} \\ \bar{\rho} - \rho &= (\mu - \bar{\mu})\, e^{\phi+\bar{\phi}} \\ \sigma &= \bar{\lambda}\, e^{\phi+\bar{\phi}} \\ \kappa &= (\bar{\alpha} - \beta + \bar{\pi} - \delta\bar{\phi} + \delta\phi)\, e^{\phi+\bar{\phi}} \\ 0 &= \epsilon - \bar{\epsilon} + D\bar{\phi} - D\phi \end{aligned}$$

Thus two neutrino fields ξ_A and $e^\phi\chi_A$ will be admitted by the same metric provided

$$\epsilon = \rho, \qquad \beta = \tau, \qquad \gamma = \bar{\gamma} \tag{2.2}$$

and provided a function ϕ can be found such that the equations

$$\alpha - 2\bar{\tau} = -\nu\, e^{\phi+\bar{\phi}}, \qquad \bar{\rho} - \rho = (\mu - \bar{\mu})\, e^{\phi+\bar{\phi}}, \qquad \sigma = \bar{\lambda}\, e^{\phi+\bar{\phi}} \tag{2.3}$$

$$\bar{\delta}\phi = \alpha - \pi, \qquad \Delta\phi = \gamma - \mu \tag{2.4}$$

$$D\phi - D\bar{\phi} = \rho - \bar{\rho}, \qquad \delta\phi = \tau - 2\bar{\pi} + \kappa\, e^{-\phi-\bar{\phi}} \tag{2.5}$$

are all satisfied. Substituting (2.2) and (2.3) into the Newman–Penrose equations (4.2 f, l, n, q) gives the condition

$$\nu(\kappa + \bar{\pi}\, e^{\phi+\bar{\phi}}) - \bar{\nu}(\bar{\kappa} + \pi\, e^{\phi+\bar{\phi}}) + \rho\bar{\mu} - \bar{\rho}\mu = 0$$

The integrability conditions for equations (2.4) and (2.5) may easily be calculated by the Newman–Penrose method but these do not lead to any simplification unless some restrictions on the neutrino field are made.

It can be seen that non-unique neutrino-gravitational fields with flux vectors in different directions belong to that class of fields for which the term $2i(\bar{\gamma} - \gamma)l_\mu l_\nu$ in (1.3) can be put equal to zero. They may have an energy-momentum tensor of any Plebanski type.

Some particular cases of these fields have been discussed by Collinson (1973) and Collinson & Morris (1973a, b) and some metrics obtained.

3. *A Rainich Type Theory for Neutrino-Gravitational Fields*

The energy-momentum tensor of a neutrino-gravitational field satisfies

$$E_{\mu\nu}l^\mu l^\nu = 0$$

where l_μ is the neutrino flux vector. We now consider the null vectors k_μ which satisfy

$$(R_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}R)k^\mu k^\nu = 0, \qquad k_\mu k^\mu = 0 \tag{3.1}$$

If the Ricci tensor is non-zero and trace-free, then there may exist a number of vectors k_μ. If the space-time admits a neutrino field then $R = 0$ and one null vector satisfying (3.1) will exist and will be parallel to the neutrino flux vector. If other vectors k_μ exist then it is possible that the space-time will admit two neutrino fields with flux vectors parallel to two k_μ. Thus the following approach to the Rainich theory of the neutrino field is suggested.

Consider the case when a space-time defines a Ricci tensor which is trace-free and has a unique vector satisfying (3.1). A tetrad of null vectors denoted by $k_\mu, N_\mu, M_\mu, \bar{M}_\mu$ may now be defined about k_μ, and their spin coefficients calculated. We now want to consider under what conditions the space-time will admit a neutrino-gravitational field. If it does then k_μ will be proportional to the flux vector and the tetrad will be related to the neutrino tetrad. In fact since the neutrino equations are invariant under the null rotation

$$l_\mu \to l_\mu, \qquad m_\mu \to m_\mu + a l_\mu$$

$$n_\mu \to n_\mu + a\bar{m}_\mu + \bar{a}m_\mu + a\bar{a}l_\mu$$

we may put

$$k_\mu = pl_\mu, \qquad N_\mu = p^{-1}n_\mu, \qquad M_\mu = e^{i\theta}m_\mu$$

Then by examining (1.3) and (1.4) we conclude that the space-time will admit a neutrino field if it is possible to find functions p and θ satisfying

$$\begin{aligned}
R_{\mu\nu}N^\mu N^\nu &= \frac{2i}{p}(\gamma - \bar{\gamma} + i\Delta\theta) \\
R_{\mu\nu}N^\mu M^\nu &= -i\, e^{i\theta}(\bar{\alpha} - \beta - \tau - i\delta\theta) \\
R_{\mu\nu}M^\mu \overline{M}^\nu &= ip(\rho - \bar{\rho}) \\
R_{\mu\nu}M^\mu M^\nu &= 2ip\, e^{2i\theta}\sigma \\
R_{\mu\nu}k^\mu M^\nu &= -ip^2\, e^{-i\theta}\bar{\kappa} \\
D\log p + iD\theta &= 2(\rho - \epsilon) \\
\delta \log p + i\delta\theta &= 2(\tau - \beta)
\end{aligned} \tag{3.2}$$

where the spin coefficients are now those of the tetrad $k_\mu N_\mu M_\mu \overline{M}_\mu$. If however the Ricci tensor is trace-free and has a number of vectors satisfying (3.1), then it is possible that any of these may be parallel to a neutrino flux vector. These vectors should first be computed and the conditions (3.2) examined for each. This gives us a routine for determining whether or not a space-time admits a neutrino field. It is, of course, possible that if the space-time does admit a neutrino field that it may be non-unique, and this possibility must be tested for using the methods of Section 2.

Finally, it must be emphasised that the tetrad has been chosen arbitrarily and may be transformed by any of

(a) $k_\mu \to k_\mu, \qquad M_\mu \to M_\mu + ak_\mu, \qquad N_\mu \to N_\mu + \bar{a}M_\mu + a\overline{M}_\mu + a\bar{a}k_\mu$

(b) $k_\mu \to k_\mu, \qquad M_\mu \to e^{i\eta}M_\mu, \qquad N_\mu \to N_\mu$

(c) $k_\mu \to bk_\mu, \qquad M_\mu \to M_\mu, \qquad N_\mu \to b^{-1}N_\mu$

In any particular calculation this arbitrariness may be used to simplify the equations.

4. *Space-Times Admitting a Neutrino Field or an Electromagnetic Field*

As an example and immediate application of the previous theory we consider under what conditions a space-time which admits a vacuum non-null electromagnetic field may also admit a neutrino field.

It is convenient to carry out this working using the Newman-Penrose formalism. The tetrad components of the Ricci tensor are denoted by Φ_{AB} ($A, B = 0, 1, 2$), where for a vacuum electromagnetic field

$$\Phi_{AB} = \Phi_A \bar{\Phi}_B$$

subject to Maxwell's equations which may be written

$$
\begin{aligned}
D\Phi_1 - \bar{\delta}\Phi_0 &= (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \\
D\Phi_2 - \bar{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 \\
\delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \\
\delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2
\end{aligned}
$$

The Ricci tensor is now already trace free and it will admit a vector satisfying (3.1) if Φ_{00} or Φ_{22} may be made zero. It will admit two vectors if they can simultaneously be made zero. In the case when only one vector satisfying (3.1) is admitted we may choose the tetrad such that $\Phi_{00} = 0$ and so

$$\Phi_0 = 0$$

k_μ is now a principal null vector of the electromagnetic field. Applying conditions (3.2) give

$$
\begin{aligned}
\Phi_2\bar{\Phi}_2 &= -\frac{i}{p}(\gamma - \bar{\gamma} + i\Delta\theta) \\
\Phi_1\bar{\Phi}_2 &= \frac{i}{2}e^{i\theta}(\bar{\alpha} - \beta - \tau - i\delta\theta) \\
\Phi_1\bar{\Phi}_1 &= -\frac{i}{2}p(\rho - \bar{\rho}) \\
0 &= -ip\, e^{2i\theta}\sigma \\
0 &= \frac{i}{2}p^2\, e^{-i\theta}\bar{\kappa}
\end{aligned}
$$

$$
\begin{aligned}
D\log p + iD\theta &= 2(\rho - \epsilon) \\
\delta\log p + i\delta\theta &= 2(\tau - \beta)
\end{aligned}
$$

which imply that $\sigma = 0 = \kappa$ and hence using the Newman-Penrose equations that $\Psi_0 = 0 = \Psi_1$.

We are now free to make a null rotation on the tetrad such that

$$\bar{\alpha} - \tau - \beta - i\delta\theta = 0$$

which can always be done provided $\rho \neq 0$—a condition which must hold if the electromagnetic field is to be non-null. It can now be seen that either $\Phi_1 = 0$, and we are considering a null field with k_μ a repeated principal null direction which is parallel to a shear-free and twise-free geodesic congruence, or $\Phi_2 = 0$. In the latter case the only non-zero components of the Ricci tensor are given by

$$\Phi_1\bar{\Phi}_1 = -\tfrac{1}{2}\, ip(\rho - \bar{\rho})$$

subject to

$$D\Phi_1 = 2\rho\Phi_1, \qquad \bar{\delta}\Phi_1 = -2\pi\Phi_1, \qquad \delta\Phi_1 = 2\tau\Phi_1, \qquad \Delta\Phi_1 = -2\mu\Phi_1$$

All these equations will be integrable provided

$$(\rho - \bar{\rho})\Delta \log p = -(\rho - \bar{\rho})(\gamma + \bar{\gamma}) - 2(\rho - \bar{\rho})(\mu + \bar{\mu}) + \Psi_2 - \bar{\Psi}_2 \tag{4.1}$$

$$\bar{\delta}\rho = 2(\bar{\tau} + \pi)(\bar{\rho} - \rho) + \rho(\alpha + \bar{\beta}) \tag{4.2}$$

$$\Delta\tau = \bar{\nu}(2\bar{\rho} - \rho) + \tau(\gamma - \bar{\gamma}) - \bar{\Psi}_3 \tag{4.3}$$

$$\Delta(\rho - \bar{\rho}) = (\rho - \bar{\rho})(\gamma + \bar{\gamma}) - \Psi_2 + \bar{\Psi}_2 \tag{4.4}$$

$$\Delta\rho + D\mu = (\gamma + \bar{\gamma})\rho - (\epsilon + \bar{\epsilon})\mu + \pi\bar{\pi} - \tau\bar{\tau} \tag{4.5}$$

$$\delta\mu + \Delta\tau = \rho\bar{\nu} - \mu(\bar{\alpha} + \beta) + \bar{\lambda}\pi + \tau(\gamma - \bar{\gamma}) \tag{4.6}$$

$$D\pi = 2(\rho - \bar{\rho})(\bar{\tau} + \pi) - (\epsilon - \bar{\epsilon})\pi \tag{4.7}$$

$$\Delta\pi - \bar{\delta}\mu = -\rho\nu - (\bar{\tau} - \alpha - \bar{\beta})\mu + \lambda\tau - (\bar{\mu} + \gamma - \bar{\gamma})\pi \tag{4.8}$$

in addition to the Newman-Penrose field equations.

In order to confirm these results we may now take the more conventional approach of taking the expression for the neutrino energy-momentum tensor (1.3) and applying to it the Rainich conditions

$$R_\mu{}^\mu = 0 \tag{4.9}$$

$$R_{\mu\nu}V^\mu V^\nu < 0 \tag{4.10}$$

for all time-like vectors V_μ,

$$R_{\mu\alpha}R^{\alpha\nu} \doteq \Lambda^2 \delta_\mu{}^\nu \tag{4.11}$$

$$S_{\kappa;\tau} = S_{\tau;\kappa} \tag{4.12}$$

where

$$S_\kappa = \Lambda^{-2} E_{\kappa\sigma\mu\nu} R^\mu{}_\alpha R^{\alpha\nu;\sigma}$$

In order to make these results immediately comparable with the previous ones it is convenient to introduce a tetrad $L_\mu, M_\mu, \bar{M}_\mu, N_\mu$ which is parallel with the neutrino tetrad according to

$$L_\mu = p l_\mu, \qquad M_\mu = e^{i\theta} m_\mu, \qquad N_\mu = p^{-1} n_\mu$$

Using the spin coefficients of this new tetrad the components of the neutrino energy-momentum tensor and Weyl's equation are identical to those given by equations (3.2) except that L_μ replaces k_μ.

Equation (4.9) is already satisfied, and it has been shown elsewhere (Griffiths & Newing, 1971a; Wainwright, 1971) that if (4.10) is satisfied then

$$\kappa = 0$$

(i.e. $R_{\mu\nu}L^\mu M^\nu = 0$), and we can put

$$\alpha - \bar{\beta} - \bar{\tau} + i\delta\theta = 0$$

(i.e. $R_{\mu\nu}N^\mu M^\nu = 0$). Also

$$A \geqslant 0 \qquad \text{and} \qquad \omega \geqslant \tfrac{1}{2}|\sigma| \qquad \text{where } \omega = \frac{i}{2}(\bar{\rho} - \rho)$$

Condition (4.11) may be satisfied by two distinct cases

(a) $\sigma = 0, \omega = 0, \Lambda = 0$. This is the case of the neutrino pure radiation field and corresponds to a null electromagnetic field. This case is being neglected here since the Rainich conditions break down in this case.

(b) $\sigma = 0, \gamma - \bar{\gamma} + i\Delta\theta = 0, \Lambda^2 = 4p^2\omega^2$. In this case there are two vectors satisfying (3.1) and the non-uniqueness mentioned in Section 2 might occur. The neutrino flux vector which is parallel to a principal null direction of the electromagnetic field defines a shear-free but twisting null geodesic congruence.

In the case (b) the energy-momentum tensor has the form

$$E_{\mu\nu} = 4p\omega(L_\mu N_\nu + N_\mu L_\nu + M_\mu \bar{M}_\nu + \bar{M}_\mu M_\nu)$$

and we obtain

$$S_\mu = 2\{(\mu - \bar{\mu})L_\mu + (\bar{\rho} - \rho)N_\mu - (\pi + \bar{\tau})M_\mu + (\bar{\pi} + \tau)\bar{M}_\mu\}$$

and (3.12) implies

$$D(\mu - \bar{\mu}) + \Delta(\rho - \bar{\rho}) = (\rho - \bar{\rho})(\gamma + \bar{\gamma}) - (\mu - \bar{\mu})(\epsilon + \bar{\epsilon}) \tag{4.13}$$

$$\begin{aligned}\Delta(\pi + \bar{\tau}) - \bar{\delta}(\mu - \bar{\mu}) = {} & (\mu - \bar{\mu})(\alpha + \bar{\beta}) - \mu\bar{\tau} - \bar{\mu}\pi - (\rho - \bar{\rho})\nu \\ & + (\bar{\pi} + \tau)\lambda - (\pi + \bar{\tau})(\gamma - \bar{\gamma})\end{aligned} \tag{4.14}$$

$$D(\pi + \bar{\tau}) + \bar{\delta}(\rho - \bar{\rho}) = (\rho - \bar{\rho})(\alpha + \bar{\beta}) + \rho\bar{\tau} + \bar{\rho}\pi - (\pi + \bar{\tau})(\epsilon - \bar{\epsilon}) \tag{4.15}$$

$$\delta(\pi + \bar{\tau}) + \bar{\delta}(\bar{\pi} + \tau) = (\bar{\pi} + \tau)(\alpha - \bar{\beta}) + (\pi + \bar{\tau})(\bar{\alpha} - \beta) \tag{4.16}$$

We also require that (4.1), (4.2), (4.3) and (4.4) be satisfied since these are the integrability conditions on the Weyl equation and the two conditions $R_{\mu\nu}N^\mu N^\nu = 0$ and $R_{\mu\nu}N^\mu M^\nu = 0$. Using these and the Newman-Penrose equations it can be shown that conditions (4.5)-(4.8) are identical to (4.13)-(4.16) and our calculations are confirmed. The most convenient way to show that (4.13)-(4.16) imply (4.5)-(4.8) is to first show that the Weyl equations imply that

$$D\Phi_{11} = 2(\rho + \bar{\rho})\Phi_{11}, \qquad \delta\Phi_{11} = 2(\tau - \bar{\pi})\Phi_{11}, \qquad \Delta\Phi_{11} = -2(\mu + \bar{\mu})\Phi_{11}$$

which are equivalent to the equations $E^{\mu\nu}{}_{;\nu} = 0$.

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Note added in proof: The subject of this paper has also been considered independently by Ludwig. His results will appear in a series of papers in *Journal of Mathematical Physics* entitled "On the geometrization of neutrino fields". His approach to the Rainich theory of neutrino fields is more detailed than that given here and the ambiguities arising here in Section 3 are avoided. I am also grateful to Dr. Ludwig for having pointed out a serious error in the original manuscript.